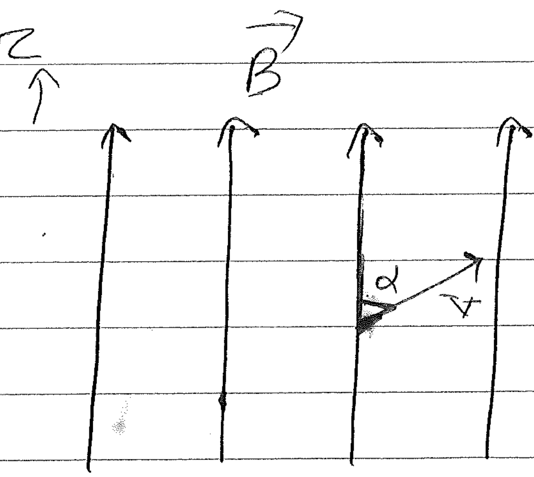
Lec 11: 09/24/2014

Synchrotron Radiation:

The energy and momentum transferred to the radiating electron by an ion's Coulomb field may also be provided by a magnetic field $\vec{B}$, which accelerates the electron according to the Lorentz force.

A non-relativistic electron's motion in the presence of a magnetic field is a superposition of a translational path with constant velocity $v_{\parallel} = v\cos\alpha$ and a circular accelerated component with speed $v_{\perp} = v\sin\alpha$:



$$m_e \frac{v_\perp^2}{r_{gyr}} = \frac{eBv_\perp}{c} \Rightarrow$$

$$r_{gyr} = \frac{v\sin\alpha}{\omega_{gyr}}$$

$$\omega_{gyr} \equiv \frac{eB}{m_e c} \simeq 1.8\times10^{7}\left(\frac{B}{1G}\right)$$

The power emitted by the electron is given by Larmor's formula:

$$P = \frac{2e^2}{3c^3}\,\omega_{gyr}^2\, v^2 \sin^2\alpha$$

It emerges as monochromatic radiation with angular frequency ω_{gyr}. It is known as "Cyclotron Radiation". The monochromatic spectrum associated with cyclotron radiation is essentially independent of the viewing angle. As we will see later, this situation will change dramatically when $v \to c$.

In the relativistic limit, it will be more convenient to move to the (instantaneous) rest frame of the electron. In this frame (shown by $'$) the electric and magnetic field components are (electron velocity in the lab frame is in the x direction);

$$E^{x'} = E^x \quad , \quad E^{y'} = \gamma\left(E^y - \frac{v}{c}B^z\right) \quad , \quad E^{z'} = \gamma\left(E^z + \frac{v}{c}B^z\right)$$

In the lab frame $\vec{E} = 0$, and $B^x = B^y = 0$. Thus:

$$E^{y'} = -\frac{\gamma v}{c} B\sin\alpha$$

Since electron is at rest in its rest frame, we have;

$$\vec{F} = -e\vec{E} \Rightarrow \vec{a}' = \frac{e\gamma v B\sin\alpha}{m_e c} = \gamma v \omega_{gyr} \sin\alpha$$

The power radiated by the electron in its rest frame is:

$$P' = \frac{2e^2}{3c^3}\gamma^2 \omega_{gyr}^2 v^2 \sin^2\alpha$$

It can be written in terms of the Thomson scattering cross section σ_T and the energy density in the magnetic field $U_B = \frac{B^2}{8\pi}$:

$$P' = \frac{4}{3}\sigma_T c\beta^2\gamma^2 U_B$$

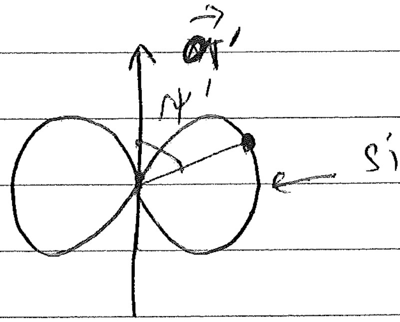
This, as we will see later, proves useful when we compare the power radiated by synchrotron radiation with that in Compton scattering.

We note that the radiated power is a Lorentz-invariant quantity since $P' = \frac{dE'}{dt}$ and $P = \frac{dE}{dt}$, where both energy and time are transformed as the 0-th component of a 4-vector.

This implies that the total power in the lab frame is:

$$P_{sync} = \frac{4}{3}\sigma_T c\beta^2\gamma^2 U_B$$

Next, we consider the frequency dependence of the synchrotron radiation. As mentioned before, the radiation pattern in the electron's rest frame is that of a dipole:



However, it will look very different in the lab frame because $v \simeq c$ in this frame. The transformation law for the angles in the two frames, ψ' and ψ, is:

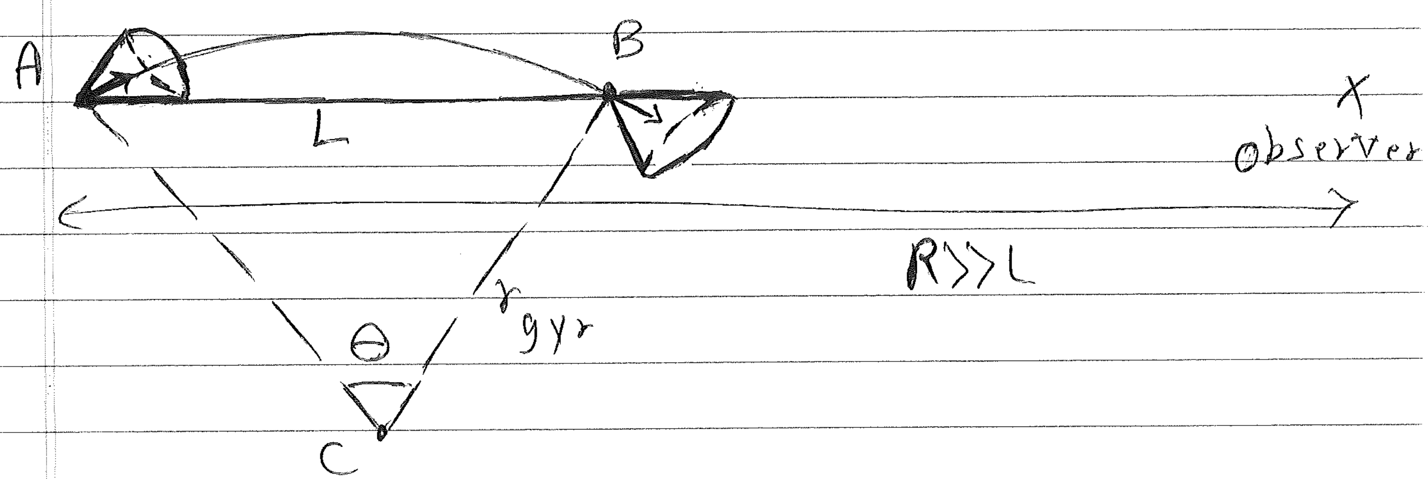
$$\sin\psi = \frac{\sin\psi'}{\gamma(1+\beta\cos\psi')}$$

For $\beta \simeq 1$ and $\gamma \gg 1$, this results in:

$$\sin\psi \approx \psi \approx \frac{1}{\gamma}$$

Thus, whereas the power is radiated nearly isotropically in the electron's rest frame, most of it is beamed into a narrow cone with half-opening angle $\approx \frac{1}{\gamma}$ in the lab frame.

Because of this beaming, the emission is pulsed every time the cone sweeps around the line of sight. Another important point has to do with the actual time interval over which the observer sees radiation:



$$t_{AB} = \frac{L}{v}$$

$$t^{o}_{AB} = t_{AB} - \frac{L}{c} = t_{AB}\left(1 - \frac{v}{c}\right)$$

↑ arrival time at the observation point

It turns out from the above discussion that $\theta \approx \frac{2}{\gamma}$, which results in:

$$t^{o}_{AB} \approx \left(1 - \frac{v}{c}\right)\frac{2}{\gamma}\frac{1}{\omega^{rel}_{gyr}}$$

We note that the relativistic angular gyration frequency ω^{rel}_{gyr} is different from ω_{gyr}, mentioned before, as we have:

$$\vec{F} = \frac{d\vec{p}}{dt} = \gamma m_e \frac{d\vec{v}}{dt} = -\frac{e}{c}\vec{v} \times \vec{B}$$

Since $\vec{E}=0$ in the lab frame, γ is constant, and can be taken out. Thus:

$$\omega^{rel}_{gyr} = \frac{eB}{\gamma m_e c} = \frac{\omega_{gyr}}{\gamma}$$

We therefore find:

$$t^{0}_{AB} \simeq \frac{1}{\gamma^2 \omega_{gyr}}$$

It is seen that the arrival time in the relativistic limit is much shorter than the orbital period:

$$T_{orbit} = \frac{2\pi}{\omega^{rel}_{gyr}} \sim \gamma^3 t^{0}_{AB}$$

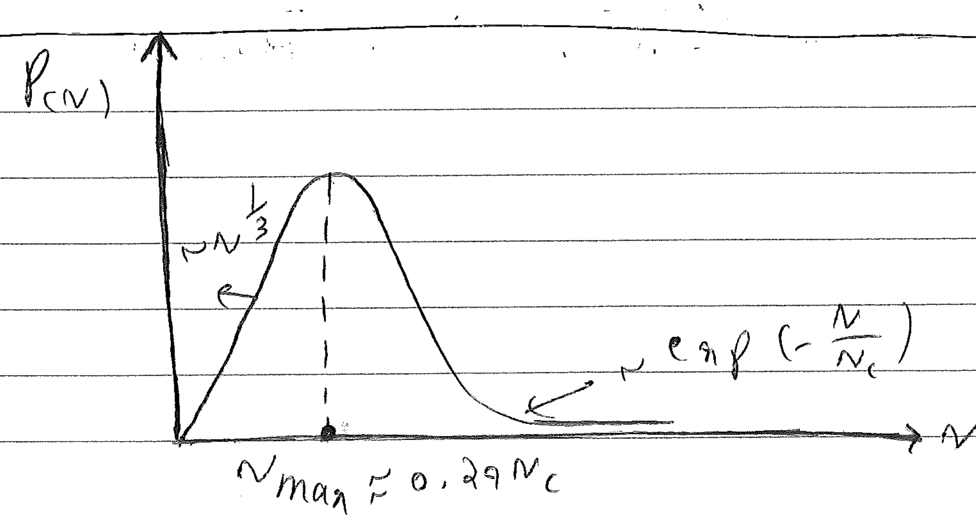
Hence the dominant frequency of radiation $\sim (t^{0}_{AB})^{-1}$ is much higher than ν^{rel}_{gyr}. In addition, because

$\omega_{gyr}^{rel} \ll \omega_{gyr}$, the spacing between adjacent frequencies is much smaller than that in the non-relativistic limit. As a result, many discrete emission lines blen together near $\gamma^2 \nu_{gyr}$ to form a continuum in the relativistic case. Taking all the Fourier components into account results in the following expression for the total power per unit frequency:

$$P(\nu)\, d\nu = \frac{\sqrt{3}\, e^3 B \sin\alpha}{m_e c^2} \frac{\nu}{\nu_c} d\nu \int_{\frac{\nu}{\nu_c}}^{\infty} K_{\frac{5}{3}}(\eta)\, d\eta$$

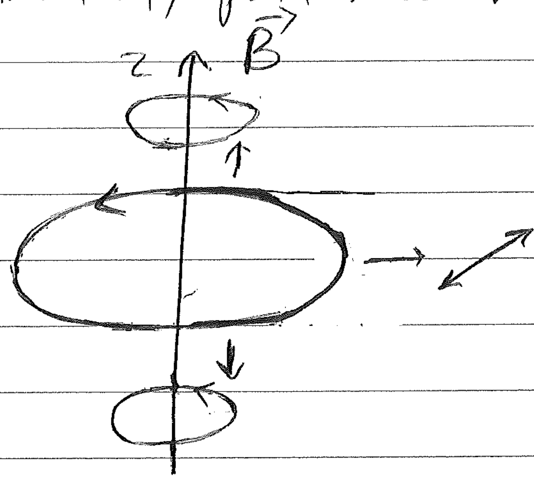
Here $K_{\frac{5}{3}}$ is the Bessel function of order $\frac{5}{3}$ and;

$$\nu_c \equiv \frac{3}{2} \gamma^2 \nu_{gyr} \sin\alpha$$



However, the use of the expression on the previous page is rather limited. Synchrotron radiation from astrophysical sources is typically produced by an ensemble of particles. This removes the pitch angle α from the expression, assuming an isotropic distribution, and one has to consider an energy distribution for the electrons. We will discuss the spectrum of synchrotron radiation for both thermal and non-thermal energy distributions later. But, first, let us consider another important characteristic of synchrotron emission, namely the polarization.

Synchrotron radiation, unlike Bremsstrahlung emission, is polarized. In the case of Bremsstrahlung the electrons accelerate in a completely random fashion, which results in unpolarized radiation. Therefore, the best one can hope to measure is the average intensity. However, synchrotron emission is polarized in a simple way. The reason being that all electrons gyrate in the same way (Clockwise or Counterclockwise, depending on the direction of $\vec{B}$). In the non-relativistic case, cyclotron emission, an observer in the plane of electron's orbit see linearly polarized radiation, while the observer at the top or bottom sees a circularly polarized radiation:



The relativistic limit is considerably different because the radiation is strongly beamed in the forward moving direction. The net result is a linearly polarized wave in this case. For a source with a uniform magnetic field, the overall degree of polarization can be as high as ~70%. In reality, we very rarely have objects in nature with a well-structured, global magnetic field. More typically, the field will have both a turbulent component and a contribution from a large scale uniform component. This reduces the overall degree of polarization in synchrotron radiation to values around a few percent.

The detection of polarized emission, usually in the radio waveband, is a powerful diagnostic indicating that the radiation is of synchrotron origin.